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FACES OF SIMPLEXES OF INVARIANT MEASURES

BY

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ABSTRACT

This paper is a direct continuation of [D2]. The main result proved here, combined with Theorem 1 of [D2] widens the variety of known possibilities of what the simplex of invariant measures in a minimal topological dynamical system can be. We show that it can be equivalent, modulo affine homeomorphism of Choquet simplexes and modulo isomorphisms of the ergodic measures playing the role of the extreme points, to any face of the simplex of invariant measures of any zero-dimensional topological dynamical system, as long as this face contains no periodic measures. In particular, this implies that any formal simplex spanned by any choice of countably many nonperiodic ergodic measures can be realized in a minimal system.

1. Introduction

By an **assignment** we understand a function $\Psi : K \to \{\text{mpt's}\}$ defined on an abstract metrizable Choquet simplex K (see, e.g., [P] for the definition of a Choquet simplex), whose "values" are measure-preserving transformations, as follows: for $p \in K$, $\Psi(p) = (X_p, \Sigma_p, \mu_p, T_p)$, where T_p is an endomorphism of a standard probability space (X_p, Σ_p, μ_p). Two assignments, Ψ and Ψ' defined on two simplexes K and K', respectively, are considered **equivalent** if there

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exists an affine homeomorphism $\pi : K \to K'$ such that $\Psi(p)$ and $\Psi'(\pi(p))$ are isomorphic (as measure-preserving transformations) for every $p \in K$.

If (X, T) is a topological dynamical system (i.e., $T : X \to X$ is a continuous self-map of a compact metric space) then the set of all T-invariant measures supported by X, endowed with the weak* topology of measures, is a Choquet simplex, and the assignment by identity $\Psi(\mu) = (X, \Sigma_B, \mu, T)$ (where Σ_B is the Borel sigma-field) is what we call the **natural assignment** of (X, T).

An assignment is called **topological** (**minimal**) if it is equivalent to a natural assignment of a topological (minimal) dynamical system. Two obvious conditions for an assignment to be topological are: (1) systems assigned to the extreme points must be ergodic, and (2) the assignment is (up to isomorphism) determined by its restriction to the extreme points via **the ergodic decomposition rule** (see [D2] for a more precise formulation). We are interested in providing further criteria allowing to determine whether a given abstract assignment is minimal (or at least topological).

The main result of [D2] asserts that any topological assignment arising from a zero-dimensional dynamical system without periodic points is also minimal. Note that **nonperiodicity** i.e., lack of periodic systems in the range (equivalently, lack of periodic points in any topological realization) is a necessary requirement for a nontrivial topological assignment to be minimal. This result has allowed us, among other things, to establish that every nonperiodic assignment on a finite set extends (by convex combinations of measures) to a minimal assignment on the simplex spanned by this set: the disjoint union of strictly ergodic realizations via Jewett-Krieger Theorem (more precisely, via its version for noninvertible systems proved by Rosenthal [R]) provides a nonperiodic zero-dimensional model. In [D2], we have postulated a conjecture that similar freedom should be valid for Choquet simplexes spanned by countable sets of extreme points. As another application of the result in [D2] we have constructed a noninvertible version of a **universal minimal system** of B. Weiss (see [W]): a minimal system whose assignment's range contains, up to isomorphism, all possible (both invertible and noninvertible) nonperiodc measure-preserving transformations.

In the meantime, I. Kornfeld and N. Ormes [K-O] have proved a beautiful theorem, implying that our aforementioned conjecture is true whenever all the systems assigned to the countably many extreme points of the simplex are (in addition to being nonperiodic) invertible. The authors are even able to construct a minimal realization of an arbitrary such assignment on a simplex K within any topological orbit equivalence class of Cantor minimal systems whose simplex of invariant measures is affinely homeomorphic to K (the affine-topological "shape" of this simplex is an invariant of the orbit equivalence relation, so this requirement cannot be skipped). The methods used by the authors are completely different from those of [D2] and rely on multitowers constructions and manipulations of the order of the floors. These orbit techniques seem impossible to be used in the noninvertible situation, also, any applications to simplexes with uncountable sets of extreme points seem rather hard. Our method, based on symbolic representations and codes, allows us to deal with nonivertible systems and uncountable extreme sets, but in turn, are completely unfit for exploration of the orbit equivalence classes. Unfortunately, neither methods allow excursions beyond dimension zero, further than some direct consequences of theorems on zero-dimensional modeling.

In this work, sticking to our symbolic methods, we push a bit further. We prove that any "nonperiodic face" of a topological zero-dimensional assignment Ψ is itself a topological zero-dimensional assignment, hence, by the result of [D2], also a minimal one. By a **nonperiodic face** of an assignment Ψ on a simplex K we mean the restriction of Ψ to a face of K (i.e., to a subsimplex whose all extreme points are extreme in K), such that the range of this restriction avoids periodic systems. We do not require that the entire assignment Ψ is nonperiodic, which will allow us to apply the theorem to the full shift on a finite or Cantor alphabet. This result permits, among other things, to establish that the conjecture formulated in [D2] is true in general, i.e., without assuming invertibility. To demonstrate the potential of the result also for simplexes with uncountably many extreme points, we describe a class of "Bernoulli assignments" hitherto not known to be minimal.

2. Marker lemmas

Throughout this paper we will be working in the following context: (X, T) is a zero-dimensional topological dynamical system, i.e., X is a zero-dimensional compact metric space and $T : X \to X$ is continuous. We are not assuming invertibility or surjectivity of T. The key tool in this work will be so-called **markers**, i.e., sets $F \subset X$ such that every orbit from a suitable set visits F infinitely many times with gap lengths in a specified range. The goal of this section is to extend X to a "markered system" $\tilde{X}$ equipped with a decreasing sequence of clopen (i.e., closed and open) marker sets, each visited by every orbit infinitely many times with gaps in a specified range. Precise description is formulated in the Lemma 2.13.

Let us begin with a variant of "Krieger's Marker Lemma" (as it is called in [B]). The reader may find an invertible expansive version in [B] and a non-invertible version in absence of periodic points in [D2]. Below we make no assumptions other than dimension zero of X.

Lemma 2.1: *For every $n \geq 1$ and $\epsilon > 0$ there exists a clopen set F and $N \in \mathbb{N}$ such that:*

(2.2) *$T^{-i}F$ are pairwise disjoint for $i = 0, 1, \ldots, n$, and*
(2.3) *$T^{-i}F$ for $i = 0, 1, \ldots, 2n$ cover the set $X \setminus T^{-N}(P_n^\epsilon)$*

where P_n^ϵ denotes the ϵ-neighborhood of the set P_n of all periodic points with periods not exceeding n.

Proof. The set P_n is closed, so replacing, if necessary, P_n^ϵ by a smaller neighborhood we can assume that P_n^ϵ is clopen. For given n every point $x \in X \setminus P_n^\epsilon$ belongs a clopen set $E_x \subset X \setminus P_n^\epsilon$ such that its $n+1$ consecutive preimages $T^{-i}(E_x)$ $(0 \leq i \leq n)$ are pairwise disjoint. Choose a finite cover $\mathcal{U} = \{U_j\}_{1 \leq j \leq m}$ of $X \setminus P_n^\epsilon$ by some of the sets E_x. The family $\mathcal{U}' = \{U_j'\} = \{T^{-nm}(U_j)\}$ (m equals the cardinality of $\mathcal{U}$), is a cover of $X \setminus T^{-nm}(P_n^\epsilon)$ with the same property that each element has pairwise disjoint $n+1$ consecutive preimages.

Define inductively

$$F_1 := U_1'$$
$$F_{j+1} := F_j \cup \left(U_{j+1}' \setminus \bigcup_{-n \leq i \leq n} T^i(F_j) \right)$$

and set $F = F_m$. Because in fact this construction involves set operations only on the sets $T^{-k}(U_j)$, with $k \geq n$, it is seen that the sets F_j $(1 \leq j \leq m)$ are clopen, and have, for every pair of integers i, i' between 0 and n, the property $T^{-i'}(T^i(F_j)) = T^{i-i'}(F_j)$. The verification of (2.2) is now straightforward. For (2.3), let $N = nm + n$ and consider a point $y \in X \setminus T^{-N}(P_n^\epsilon)$. Because $T^n(y) \in X \setminus T^{-nm}(P_n^\epsilon)$, there is $1 \leq j \leq m$ such that $T^n(y) \in U_j'$. If $j = 1$ then $T^n(y) \in F_1 \subset F$, hence $y \in T^{-n}(F)$. Otherwise, we can write $T^n(y) \in U_{j-1+1}'$ and then either $T^n(y) \in F_j$ which again implies $y \in T^{-n}(F)$, or else it must be that $T^n(y) \in \bigcup_{-n\le i\le n} T^i(F_{j-1})$. But in the last case, $T^n(y) \in T^i(F_{j-1})$ for some $-n \le i \le n$, hence $y \in T^{i-n}(F_{j-1}) \subset T^{i-n}(F)$, where $-2n \le i-n \le 0$. This completes the proof. ∎

If (X,T) is represented as a subshift system over some (compact) alphabet A (i.e., $X \subset A^{\mathbb{N}_0}$ and $T(x)_i = x_{i+1}$ for $x = (x_i)_{i\in\mathbb{N}_0} \in X$), and the marker set F is clopen, it is customary to mark the visits of the orbit of a point x in F at times $n_1, n_2, \dots$ by placing some kind of extra symbol (star) above the coordinates $n_1, n_2, \dots$ in the sequence representing x. Because F is clopen, such procedure produces a topologically conjugate representation of the system (X,T) (now with the enhanced alphabet $A \times \{\emptyset, *\}$).

Lemma 2.4: *Let (X,T) be a zero-dimensional dynamical system. For every $n \ge 1$ there exists a set G such that:*

(2.5) *$T^{-i}G$ are pairwse disjoint for $i = 0,1,\dots,n$,*
(2.6) *$T^{-i}G$ for $i = 0,1,\dots,2n$ cover the set $X \setminus AP_{n!}$, and*
(2.7) *$\delta G \subset AP_{n!}$,*

where $AP_{n!}$ denotes the set of points asymptotic to points from $P_{n!}$ and δG denotes the boundary of G.

Proof. In this proof we find the "symbolic" setup (with cylinder sets) more convenient than dealing with general clopen sets and their preimages. The zero-dimensional system (X,T) can be represented as an inverse limit of subshifts over finite alphabets

$$\overleftarrow{\lim_{i\to\infty}}(X_i, S_i),$$

i.e., each $x \in X$ has a "matrix" form: $X \ni x = [x_{i,j}]_{i\in\mathbb{N}, j\in\mathbb{N}_0}$ where, for each i, $x_{i,j}$ ranges over a finite set Λ_i, and the map on X is $T(x) = [x_{i,j+1}]$.

Fix some $n \in \mathbb{N}$. By Lemma 2.1 applied to (X_i, S_i), each row x_i of each matrix x comes equipped with a system of stars with the following properties:

(2.8) the stars appear at distances not smaller than $n+1$,
(2.9) if a star at position j is followed by a gap (with no stars) longer than $2n$ then the block in x_i between positions $j + N_i$ and $j + N_i + 2n + 1$, is periodic with a period not exceeding n.

(Lemma 2.1 ensures that such gap implies that the point $S_i^j(x_i)$ is in $S_i^{-N_i}(P_n^\epsilon)$. Because the alphabet is finite, we may choose ϵ small enough, so that this will imply (2.9).) We shall now define an intermediate marker set G', as follows: for a matrix x observe the trapezoidal regions of coordinates $\Delta_k = \{(i,j) : i+j \leq 2n+k, i \leq k\}$. We classify x to G' if and only if there exists k, such that a star appears exactly at the position (k,n) and it is unique within Δ_k (see Figure 1 below for Δ_4 $(n=3)$). We claim the following:

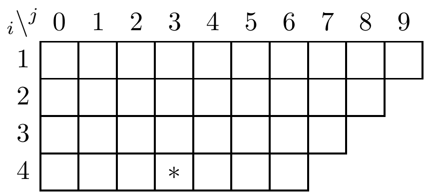

Figure 1

(2.10) $T^{-i}G'$ are pairwse disjoint for $i = 0, 1, \ldots, n$,

(2.11) $T^{-i}G'$ for $i = 0, 1, \ldots,$ cover the set $X \setminus AP_{n!}$, and

(2.12) $\delta G' \subset AP_{n!}$.

For (2.10) imagine a matrix $x \in G'$ visiting G' at a time $0 < m \leq n$. There would have to be a star in x at (k,n) for some k, and another one at $(k', n+m)$ for some k' such that the first one is unique in Δ_k, and the other is unique in the region $\Delta_{k'}$ shifted by m units to the right. If $k \leq k'$ then the star at (k,n) is visible in the shifted $\Delta_{k'}$, while if $k > k'$ then the star at $(k', n+m)$ is covered by Δ_k, so both cases are eliminated.

For (2.11) first observe that a row with only finitely many stars is eventually periodic with a period not larger than n: it cannot happen that for some j, the block of x_i between the positions j and $j+2n+1$ is periodic with a period $p \leq n$ and the block between $j+1$ and $j+2n+2$ is periodic with some other period $p' \leq n$ but not with p. Then note that a matrix x whose each row is eventually periodic with a period not exceeding n is asymptotically periodic with a period at most $n!$. For x not of such form, let k be the smallest index of a not eventually periodic row. The kth row contains infinitely many stars, while all rows with lower indexes (if $k > 1$) contain at most finitely many of them. Thus, we can find an $i > 0$ such that $T^i(x)$ has a star at the position (k,n) and no stars in rows with smaller indexes. By (2.8) and by the shape of Δ_k, there will be no other stars in the kth row of $T^i x$ within Δ_k. Thus the star at (k,n) will be unique in this region, classifying $T^i x$ to G'.

Finally, observe that the algorithm for classifying x to G' (or to its complement) is decisive whenever the region Δ_k covers at least one star. Also note that these regions grow with k and eventually cover the whole domain $\mathbb{N} \times \mathbb{N}_0$. Now, if a point has a row which is not eventually periodic with a period at most n, then it has a star somewhere, and thus the algorithm stops at a finite k, hence requires examining only finitely many coordinates (those within Δ_k). This implies continuity of the characteristic function of G' at x, so that x is not in the boundary, as required in (2.12).

It remains to define the desired set G. This will be done using the G'-markers (obtained at times of visits to G'). Namely, we classify an x to G if and only if the following happens: the smallest $m > 0$ with $T^m(x) \in G'$ is divisible by $n+1$. Since every point, unless it is asymptotically periodic with a period not exceeding $n!$, has a G'-marker at a positive position (apply (2.11) to $T(x)$), and finding this marker requires examining only finitely many coordinates, the classification of such a point to G or to its complement also requires examining only finitely many coordinates. So, like for G', the boundary of G is contained in $AP_{n!}$. Finally, we need to observe that every point $x \notin AP_{n!}$ will receive infinitely many G-markers appearing at distances ranging between $n+1$ and $2n+1$. Clearly, every such point has infinitely many G'-markers. Consider two consecutive G'-markers, at positions $j' < j$. It is not hard to see that the G'-marker at position j "produces" G-markers periodically, at positions $j-(n+1), j-2(n+1), j-3(n+1), \ldots$, as long as they are right of j'. Now, the G'-marker at j' does not produce a G-marker at its own position j', only at $j'-(n+1)$ (and to the left), so that a gap of size between $n+1$ and $2n+1$ is maintained. ∎

LEMMA 2.13: *Let (X,T) be zero-dimensional and such that the set of points not asymptotic to periodic points is dense. Then there exists a zero-dimensional extension $(\tilde{X},\tilde{T})$ of (X,T) which is 1-1 except on points asymptotic to periodic points in X, and such that $\tilde{X}$ admits a descending sequence of clopen marker sets $\tilde{G}_t$ ($t \geq 1$) each visited by every orbit infinitely many times with gaps ranging between p_t and $3p_t$, where p_t is a rapidly growing to infinity sequence of positive integers.*

Proof. Modulo the previous lemma, the rest of the proof relies on standard marker techniques. We will sketch it only briefly, skipping the detailed verification of several minor claims. Let $(n_t)_{t\in\mathbb{N}}$ be a rapidly increasing sequence of natural numbers. The marker set of Lemma 2.4 obtained for n_t will be denoted by G_t. We will abbreviate G_t-markers as t-markers. It is easy to modify the sets G_t to obtain new sets G'_t such that $G'_{t+1} \subset G'_t$ for every t: proceeding inductively on t, a point belongs to G'_{t+1} if it is in G'_t and this is the last visit in G'_t before (or at) a visit in G_{t+1}. Roughly speaking, a new $t+1$-marker is obtained by moving an old $t+1$-marker to the nearest new t-marker on the left. Because, to determine each new marker we never use the information left from it, this procedure is shift invariant even on one-sided sequences. The modified sets G'_t still have their boundaries contained in $AP_{n_t!}$, and all other orbits will visit them infinitely many times with gaps in a slightly enhanced range: assuming that n_{t+1} is much larger than n_t, the new gaps will still range between, say $\frac{3}{4}n_{t+1} =: p_t$ and $\frac{9}{4}n_{t+1} = 3p_t$.

We can now visualize all the (new) markers in one additional row, say, row number zero, where, at coordinate n we put the maximal integer t such that $T^n(x) \in G'_t$ (or 0 if $T^n(x) \notin G'_1$). In other words, the marker t indicates the simultaneous occurrence of the s-markers for $s = 1, 2, \ldots, t$. The points which visit the intersection of all sets G'_t will receive the marker ∞; this may happen along an orbit only once. One has to realize that because the sets G'_t are not clopen, by visualizing the markers we will change the topology of our space. We can now compactify it, by taking closure in the shift space in which the symbols t are endowed with the topology of a sequence converging to the point ∞. In such a compactification, the points whose orbits visit the boundaries of the sets G'_t (all such point belong to AP, the set of points asymptotic to periodic points) will split, receiving all possible configurations of markers resulting from approaching sequences of other points. Because the property *the gaps between occurrences of the t-markers (integers t or larger in the zero row) range between p_t and $3p_t$* is closed, and possessed by a dense set of points, it will pass to all points obtained from the split points. Note, that there is no ambiguity caused by the additional points in the closure of the union of the boundaries of G'_t which are not in AP: because they lie in the intersection of the interiors of all the sets G'_t, they will always receive the marker ∞. Such compactification $\tilde{X}$ (with $\tilde{T}$ denoting the shift map on $\tilde{X}$) factors onto X by simply deleting the zero row and this factor map is injective except on the preimage of AP. A point belongs to $\tilde{G}_t$ if and only if it has (in the row number zero) a marker t or larger at the coordinate zero. So defined sets $\tilde{G}_t$ are obviously clopen in $\tilde{X}$. ∎

Notice that there are no periodic points in the extension $\tilde{X}$.

3. The simplex lemma

As we have already mentioned, a zero-dimensional dynamical system (X, T) can be represented as an inverse limit of subshifts over finite alphabets. Viewing the columns in the resulting matrix representation as elements of the Cantor set $\mathfrak{C}$ we can replace the matrix by only one row with "symbols" in $\mathfrak{C}$, so that (X, T) becomes a subsystem of $\mathfrak{C}^{\mathbb{N}_0}$. (In this manner, $\mathfrak{C}^{\mathbb{N}_0}$ with the shift map becomes a universal system containing all zero-dimensional systems as invariant subsets.) Now, the extended system $\tilde{X}$ of Lemma 2.13 becomes a subsystem of the "universal system with markers" $\tilde{\mathfrak{C}}^{\mathbb{N}_0}$ with the alphabet $\tilde{\mathfrak{C}} := \{0, 1, 2, \ldots, \infty\} \times \mathfrak{C}$. (We picture each element as two rows: the zero row containing the markers and row number one with the symbols from $\mathfrak{C}$; there are no restrictions on the occurrences of the markers in the zero row of this large system.) For technical reasons, in the representation of $\tilde{X}$ we introduce one modification: we "double" each marker $t > 0$ in the zero row by repeating it at the preceding position (the one neighboring on the left). We may assume that $p_1 \geq 3$, so such doubling causes no collisions of markers and is reversible. Obviously, this procedure is shift invariant and leads to a conjugate representation of $\tilde{X}$.

By a **block** B of length n we will mean, a finite sequence

$$B = \langle b_0, b_1, \ldots, b_{n-1} \rangle \in \tilde{\mathfrak{C}}^n.$$

To every such block we can associate the "cylinder set" $[B] \subset \tilde{\mathfrak{C}}^{\mathbb{N}_0}$ defined as $\{x : x_i = b_i, i = 0, 1, \ldots, n-1\}$. Unlike in symbolic dynamics, this cylinder is not clopen, only closed. To each block B we can also associate the shift invariant measure μ_B supported by the periodic orbit of the point obtained as the infinite concatenation $BBB\ldots$. Notice that even if the block B appears in $\tilde{X}$, μ_B is never supported by $\tilde{X}$, but it belongs to the simplex of all invariant measures of the universal system with markers. In this large simplex we fix some metric (denoted as dist) which agrees with the weak* topology of measures. In this manner we can measure the distance between invariant measures and blocks: $\mathrm{dist}(B, \mu)$ stands for $\mathrm{dist}(\mu_B, \mu)$.

Definition 3.1: By a t-**block** we shall mean any block of length between p_t and $3p_t$ with exactly two integers larger than or equal to t in the zero row: in the leftmost and rightmost entries of this block.

Note that every element of $\tilde{X}$ decomposes in a unique way as a concatenation of t-blocks, of which the first one may be "incomplete", i.e., truncated on the left. The goal of doubling the markers was to ensure that cylinders associated to two different t-blocks are always disjoint.

LEMMA 3.2: *Let $\tilde{K}$ be a face in the simplex $\tilde{\mathcal{P}}$ of the invariant measures of $\tilde{X}$. Fix $\delta > 0$ and $\epsilon > 0$. For $t \in \mathbb{N}$, let A_t denote the union of all (closed) cylinders corresponding to the t-blocks B such that* $\operatorname{dist}(B, \tilde{K}) > \delta$. *Then there exists t such that $\mu(A_t) \le \epsilon/p_t$ for every $\mu \in \tilde{K}$.*

Proof. We will use several times the following fact on the weak topology of measures: If A is closed (open) then the function $\mu \mapsto \mu(A)$ is an upper (lower) semi-continuous function of the measure. Recall that we have a natural continuous map on the set $\mathcal{M}(\tilde{\mathcal{P}})$, of all probabilities ξ defined on the simplex $\tilde{\mathcal{P}}$ ($\mathcal{M}(\tilde{\mathcal{P}})$ is also equipped with the weak* topology), onto $\tilde{\mathcal{P}}$ by **barycenter**, i.e.,

$$\xi \mapsto \int \mu \, d\xi(\mu).$$

(The above map restricted to measures ξ supported by the ergodic measures becomes a bijection, whose inverse $\mu \mapsto \xi$ is the familiar **ergodic decomposition**.)

If now F is a closed subset of $\tilde{\mathcal{P}}$, then, by upper semi-continuity, the set of probabilities ξ with $\xi(F) \ge \epsilon$ is closed (hence compact). So its image by the barycenter map, i.e., the set of measures μ which admit a decomposition ξ (not necessarily supported by the ergodic measures) with $\xi(F) \ge \epsilon$, is also compact. The complement set (denote it by V) is hence open.

Let us return to the assumptions of the lemma. For each real-valued continuous function f on $\tilde{\mathfrak{C}}^{\mathbb{N}_0}$ the integral $\int f \, d\mu_B$ reacts continuously to the change of the values b_i defining B. Thus, if $\operatorname{dist}(B, \tilde{K}) > \delta$ (which can be determined using integrals of finitely many continuous functions) then the same holds for B replaced by any block sufficiently close to B. Also note that a block sufficiently close to a t-block is again a t-block. This implies that A_t is an open set. As a consequence, the function $\mu \mapsto \mu(A_t)$ is lower semi-continuous on invariant measures. Let now F denote the (closed) complement of the open δ ball around $\tilde{K}$ (all measures μ_B accounted in the definition of A_t are in F). By the discussion in the preceding paragraph, the set of all invariant measures such that in any decomposition the contribution of the measures from F is smaller than

ϵ, is an open set V, obviously containing $\tilde{K}$ (because $\tilde{K}$ is a face, any ξ with barycenter at $\tilde{K}$ is supported by $\tilde{K}$, so the contribution of F is zero). Let $\eta > 0$ be the radius around $\tilde{K}$ contained in V. It is elementary to see (compare Fact 2(3) in [D-S]) that if t is large enough then for any concatenation of t-blocks, $C = B_1 B_2 \cdots B_m$, the measure μ_C is $\eta/2$-close to the average

$$\overline{\mu_C} = \sum_{i=1}^{m} \frac{|B_i|}{|C|} \mu_{B_i}.$$

It suffices to prove the assertion for each ergodic measure $\mu \in \tilde{K}$. Suppose that $\mu(A_t) > \epsilon/p_t$ for such a μ. Then, by lower semi-continuity, $\nu(A_t) > \epsilon/p_t$ for all ν in some ζ-ball around μ. We may choose $\zeta < \eta/2$. By a standard fact in ergodic theory (existence of generic points), in this ball, there is a measure of the form μ_C for some long block C occurring in $\tilde{X}$. We may as well assume that C is a concatenation of t-blocks. Now, $\mu_C(A_t) = r/|C|$, where r is the count of the t-blocks B_i with $\operatorname{dist}(B_i, \tilde{K}) > \delta$ in the concatenation defining C (the markers prevent other occurrences of t-blocks in C). Since $\mu_C(A_t) > \epsilon/p_t$, we obtain $r > (\epsilon|C|)/p_t$, which means that the sum of coefficients which the selected measures μ_{B_i} receive in the average measure $\overline{\mu_C}$ (which is certainly at least $rp_t/|C|$) is bigger than ϵ. A contradiction, because the average measure is contained in V (its distance to μ is smaller than $\zeta + \eta/2 < \eta$), so the contribution of these measures μ_{B_i} must be smaller than ϵ (because they are in F). ∎

4. The main theorem

THEOREM 4.1: *Let (X, T) be a zero-dimensional dynamical system, and let K be a face in the simplex $\mathcal{P}$ of the invariant measures of (X, T). Assume that K contains no periodic measures. Then there exists another zero-dimensional dynamical system (Y, S), whose natural assignment is equivalent to the identity assignment on K.*

Before the proof we formulate the corollary important for minimal systems. It follows by composing Theorem 4.1 and Theorem 1 of [D2].

COROLLARY 4.2: *Let K be a face of the simplex of invariant measures for a zero-dimensional dynamical system. Assume that K contains no periodic measures. Then there exists a Cantor minimal system, whose natural assignment is equivalent to the identity assignment on K.*

Proof of Theorem 4.1. Because all invariant measures in K are supported by points not asymptotic to periodic ones, we can replace X by the closure of $X \setminus AP$. This will allow us to use Lemma 2.13. Fix a summable sequence of positive numbers ϵ_t ($t \in \mathbb{N}$). We begin by passing to the extension $\tilde{X}$ represented as the shift on sequences over the alphabet $\tilde{\mathfrak{C}}$ (already incorporating the markers $t \in \{0, 1, \dots, \infty\}$). Below every such sequence, (which throughout this proof will be considered a single row number one) we reserve room for further rows (numbered $2, 3, \dots$), which at the initial moment remain empty. Elements of such form (belonging to $(\tilde{\mathfrak{C}} \cup \{\emptyset\})^{\mathbb{N} \times \mathbb{N}_0}$) will be called "matrices". The metric on such matrices is defined so that rows below number t contribute to the distance by an ignorably small fraction of ϵ_t.

The simplex $\tilde{\mathcal{P}}$ of invariant measures of $\tilde{X}$ maps continuously onto $\mathcal{P}$, injectively on nonperiodic measures. Let $\tilde{K}$ denote the lift of K. Because K contains no periodic measures, the identity assignments on K and on $\tilde{K}$ are equivalent. Thus, it suffices to construct a system (Y, S) whose assignment is equivalent to the identity on $\tilde{K}$.

The proof has the following framework: By inductive application of continuous codes ϕ_t, we will perform a sequence of deformations (by the maps Φ_t conjugated to ϕ_t) of the entire simplex $\tilde{\mathcal{P}}$ toward $\tilde{K}$, which leave $\tilde{K}$ almost invariant. In the limit system only the measures originating from $\tilde{K}$ will remain unchanged (up to an isomorphism), while the other will "sink" in the limit image of $\tilde{K}$.

Great deal of the complication results from not assuming invertibility; a shift-invariant code on one-sided sequences must not refer to the information stored left from the output coordinate. The simplest idea for a code moving all invariant measures toward $\tilde{K}$ would be, to select a collection of t-blocks close to $\tilde{K}$ in the metric dist (call them "close"; all other t-blocks are "distant"), and in every element of the shift space to successively replace (in an injective way) the distant t-blocks by the close ones. In fact, such a code can be successfully applied in the invertible case. However, because for all but the first coordinate in a t-block, part of the information whether it is distant or close is contained left from this coordinate, such an algorithm cannot be applied to one-sided sequences. Instead, we will adopt a different strategy: we will group the t-blocks in finite "packages". Then we will scan every package from right to left checking every t-block for its "type". Once a distant t-block is detected, we leave it alone, but from now on we replace all the remaining t-blocks in this package (left from the detected block), regardless to their types, by the close ones. This procedure guarantees, that after the code is applied there are definitely more close t-blocks than distant ones. On the other hand, because in points belonging to the supports of the measures from $\tilde{K}$, most of the t-blocks are close (and hence there are many packages consisting entirely of close t-blocks), a large fraction of the t-blocks will not be affected at all. For this reason, the codes will converge almost surely at such points. Difficulties related to the fact that the length of the replacing concatenation of close t-blocks need not match that of the replaced concatenation will be solved by using a "cutting algorithm" introduced in [D2].

By choosing a subsequence of the marker sets, we may assume that the assertion of Lemma 3.2 is fulfilled for even indices $2t$ with $\epsilon = \delta = \epsilon_t^2$. We may arrange that the bounds p_{2t+1} and $3p_{2t+1}$ on the lengths of the $(2t+1)$-blocks equal p_{2t}/ϵ_t and $3p_{2t}/\epsilon_t$, respectively. From now on we change the terminology: the $2t$-blocks will be called t-blocks, while the $(2t+1)$-blocks will be called **packages of t-blocks**. The markers $2t$ are changed to t while $2t+1$ is replaced by t with some kind of indicator "end of package", say it looks like $t|$. The end of a t'-block ($t' > t$) is automatically an end of package of t-blocks. Every package concatenates between $1/(3\epsilon_t)$ and $3/\epsilon_t$ component t-blocks.

The map (and code) number one is the identity. Suppose we have defined a code ϕ_t (and its conjugate map Φ_t on measures), such that for each $x \in \tilde{X}$, $\phi_t(x)$ is still a concatenation of packages of t-blocks with the same length constraints as for x. Assume that in each image $\phi_t(x)$ the rows with indices larger than t remain empty, while the contents of the original one row of x is stored in the "bottom line" of $\phi_t(x)$, i.e., in the line containing in each column the last nonempty position. We can now describe the construction of the code transforming $\phi_t(x)$ to $\phi_{t+1}(x)$. In other words, ϕ_{t+1} will be a composition of ϕ_t with the code described below.

A t-block appearing in $\phi_t(\tilde{X})$ will be called "regular". A regular t-block B is "close" if $\mathrm{dist}(B, \Phi_t(\tilde{K})) \leq \epsilon_t$. Otherwise it is called "distant". We begin by fixing one close t-block B_0. The existence of such an object is guaranteed whenever the minimal length p_t of the t-blocks is sufficiently large; this condition we can easily satisfy. If the leading marker of B_0 is larger than t we replace it by t; this changes the distances insignificantly, so B_0 remains close.

Now, in every package of t-blocks in x, starting from the right end we check every component t-block. A package whose all components are close will pass to $\phi_{t+1}(x)$ unchanged (except a possible minor modification of the right end, which we describe in a moment). The first detection of a distant t-block triggers the replacement mechanism: the remaining part of the package left from that distant t-block is replaced by periodic repetitions of B_0. At the same time, we copy the contents of the bottom line of the replaced section into the (so far empty) row number $t+1$. Here we encounter further complications: the last inserted copy of B_0 may be cut at a random place. We need some control over the length of the "incomplete B_0", otherwise such blocks from several steps could accumulate in a long string violating syndetic occurrences of some smaller blocks. To achieve this control, we move the "end of package" indicator (or any marker $t' > t$ if it was there) one t-block further to the left and then we carefully choose a place where we terminate the insertion somewhere within this added t-block (see Figure 2). The description of the "cutting algorithm" which precisely determines this place will be provided later. This last modification clearly affects the following package (trims it by one t-block on the right). Because the packages must be coded in the order from left to right, the possible trimming happens after the testing and coding. The trimming is the only modification (mentioned before) which may affect a package containing no distant t-blocks. The packages of t-blocks and t'-blocks for $t' > t$ may, in this step, become shorter or longer by at most q_t. We agree that the bounds on the lengths were chosen with sufficient tolerance, so that the changed lengths remain within the same bounds. This ends the inductive construction.

It is important that the above code is applicable to one-sided sequences: even if the initial package is incomplete, we can still encode its "visible" part in a way which does not depend upon the "invisible" part.

The above code introduces in $\phi_{t+1}(x)$ some "irregular" t-blocks at the cutting places. Every such block inherits its left and right parts from two different regular t-blocks. Similarly, for each $s < t$, the same cutting produces irregular s-blocks. By obvious induction, every $\phi_{t+1}(x)$ is a concatenation of regular and irregular s-blocks (perhaps with contents added in rows $s+1, \ldots, t+1$). The cutting algorithm is constructed to ensure that in $\phi_{t+1}(x)$

(4.3) the lengths of all irregular s-blocks do not exceed $6p_s$, and

(4.4) there are at most two irregular s-blocks in every package of s-blocks.

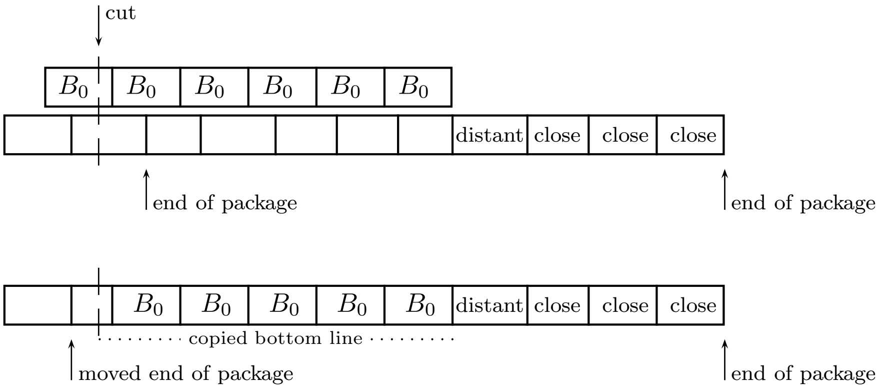

Figure 2. A package of t-blocks before and after modifications.

(We will prove this after the description of the algorithm.)

We will now show that the maps Φ_t converge uniformly on $\tilde{K}$. In step $t+1$ each measure $\mu \in \tilde{K}$ is moved from $\Phi_t(\mu)$ to $\Phi_{t+1}(\mu)$. It suffices to estimate the distances in consecutive moves by a summable series. Now Lemma 3.2 comes into play. It asserts, that for every $\mu \in \tilde{K}$ the value of $\Phi_t(\mu)(A_t)$ is smaller than $\epsilon_t^2/p_t = (3\epsilon_t^2)/q_t$ (the definition of A_t now involves $\Phi_t(\tilde{K})$ rather than $\tilde{K}$). This implies that for each x in the support of μ, at most a fraction of $3\epsilon_t^2$ of the coordinates of the sequence $\phi_t(x)$ (in terms of upper density) fall into the distant t-blocks. Thus, at most $9\epsilon_t$ of the coordinates fall into packages which contain at least one distant t-block. This implies that the modifications in passing to the image by ϕ_{t+1} will affect at most $12\epsilon_t$ of coordinates (we added $3\epsilon_t$ to include the possibly destroyed rightmost t-blocks in the packages built entirely of close t-blocks). Thus, for every continuous function f (defined on matrices) with values between 0 and 1, depending on k coordinates (columns), the difference $|\int f\, d\Phi_t(\mu) - \int f\, d\Phi_{t+1}(\mu)|$ is at most $12k\epsilon_t$. With an appropriate choice of the metric for the weak* topology, this implies that

$$\operatorname{dist}(\Phi_t(\mu), \Phi_{t+1}(\mu)) \le \epsilon_t$$

and the uniform convergence is proved. We denote by Φ the limit map on $\tilde{K}$.

Let Y be the limit space of the images $\phi_t(\tilde{X})$,

$$Y = \bigcap_{s=1}^{\infty} \overline{\bigcup_{t=s}^{\infty} \phi_t(\tilde{X})}.$$

This is a shift-invariant closed set. Assigning S to be the shift map restricted to Y we complete the definition of (Y,S). It remains to verify the assignment of this system.

At first we will show that the simplex of invariant measures of (Y,S) coincides with $\Phi(\tilde{K})$. Consider a package B_s of s-blocks occurring in Y. It is a limit, as $t\to\infty$, of packages $B_s^{(t)}$ of s-blocks occurring in the systems $\phi_t(\tilde{X})$. Every such package consists of close s-blocks, at most one distant s-block and at most two irregular s-blocks of at most doubled length. Again, with an appropriate choice of the metric for measures this implies that the measures $\mu_{B_s^{(t)}}$ are at most, say, $2\epsilon_s$ away from $\Phi_s(\tilde{K})$. Because convergence of blocks implies convergence of their periodic measures, and because every ergodic measure on Y is approximated by the measures μ_{B_s} (with s increasing), it follows that all invariant measures of (Y,S) are contained in the limit set $\Phi(\tilde{K})$. The other inclusion is true even in a more general context: for any sequence of closed invariant subsets Y_t in a topological dynamical system (X,T), and a weakly* convergent sequence of measures μ_t, each supported by Y_t, respectively, the limit measure μ is supported by a subset of the limit set $\bigcap_{s=1}^\infty \overline{\bigcup_{t=s}^\infty Y_t}$: this follows from lower semi-continuity of the map assigning to a measure its topological support.

The next claim which we will prove is the following:

(4.5) for every $\mu\in\tilde{K}$ μ-almost every x has the property that each coordinate is modified by only finitely many codes ϕ_t.

For proving of this claim, consider the set $X_t\subset X$ of points whose zero coordinate is affected by the modifications of the code ϕ_{t+1}. As we already know, the upper density of affected coordinates in μ-almost every x (for $\mu\in\tilde{K}$) is not larger than $12\epsilon_t$, so, $\mu(X_t)<12\epsilon_t$. The claim now follows immediately from summability of the series ϵ_t, because a matrix x has a column number n modified infinitely many times if it belongs to the sets $\tilde{T}^{-n}(X_t)$ for infinitely indices t.

The above proved claim (4.5) has an important consequence: the maps ϕ_t converge on a set $X'\subset\tilde{X}$ such that $\mu(X')=1$ for every $\mu\in\tilde{K}$. The limit map $\phi: X'\to Y$ is a Borel-bimeasurable injection: the original $x\in X'$ is "memorized" in the (well-defined) bottom line of $\phi(x)$. It is elementary, that the conjugate maps Φ_t converge to the map conjugate to ϕ, which hence coincides with Φ. This implies that for every $\mu\in\tilde{K}$, the system $(\tilde{X},\mu,\tilde{T})$ is isomorphic to $(Y,\Phi(\mu),S)$ via the map ϕ, moreover, Φ is injective on $\tilde{K}$. We already know that it is onto all invariant measures of Y, and that it is continuous (hence a homeomorphism). Thus the natural assignment of (Y, S) (i.e., the identity on $\Phi(\tilde{K})$) is equivalent to the identity on $\tilde{K}$.

It remains to describe the cutting algorithm. It is almost identical as described in [D2], but because of a few changed details we repeat the description. It will be established in t steps enumerated decreasingly $t, t-1, t-2, \ldots, 1$. We begin by placing a temporary mark $*$ $8p_t$ positions left from left end mark of the considered package of t-blocks in $\phi_t(x)$. This concludes the step t. In each step $s < t$ we find the nearest "end of package" marker $s|$ (it can also have the form t', $(s < t' \leq t)$; from now on we skip reminding the reader of this) in $\phi_t(x)$ left from (or at) the mark $*$, and then we find the nearest marker $s|$ in the inserted blocks B_0 left from there. If the distance between these two markers $s|$ is at least $16p_s$ then we move the $*$ mark to the position $8p_s$ units left from the considered marker $s|$ in $\phi_t(x)$, otherwise we put it $8p_s$ units left from the considered marker $s|$ in the inserted blocks B_0 (see Figure 3 below).

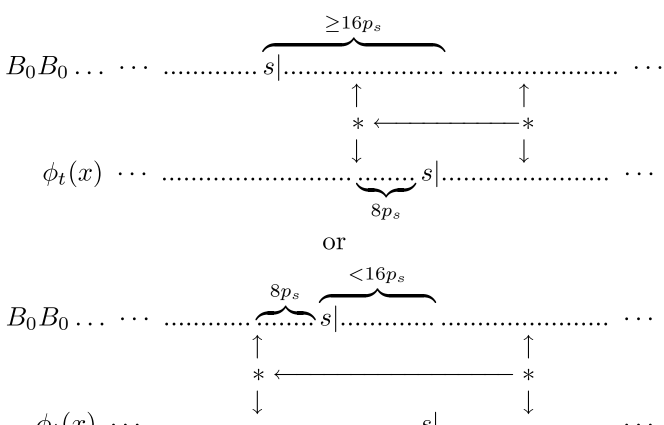

Figure 3

Then we pass to step $s-1$. The position of the mark $*$ after step 1 is where we cut.

We need to prove statements (4.3) and (4.4). Assuming them for codes $\phi_{t'}$ with $t' \leq t$ we need to examine only the "new" irregular s-blocks introduced by the code ϕ_{t+1}. Initially the mark $*$ is at least $8p_t$ positions away from the ends of any packages of t-blocks in $\phi_t(x)$ (and in the inserted blocks B_0 there are no such packages). In the following steps $t' < t$ the mark is moved by at most $(3p_{t'})/\epsilon_{t'} + 24p_{t'}$, (recall that $(3p_{t'})/\epsilon_{t'}$ is the maximal length of a package of t'-blocks) and after that move the distances from the mark $*$ to the nearest markers $t'|$ in $\phi_{t'}(x)$ and in the inserted blocks B_0 are at least $8p_{t'}$ on both sides. Further moves are small compared with $p_{t'}$, hence eventually the cutting place falls, say, $7p_{t'}$ away from the markers $t'|$ in both $\phi(x)$ and in the blocks B_0. Now, any irregular s-block created in an earlier code $\phi_{t'+1}$ $(s \le t' < t)$ lies near the end of a package of t'-blocks (it is inside the leftmost t'-block in such package, which is irregular and whose length, by inductive assumption is at most $6p_{t'}$). Thus the "new" irregular s-block is "made" from two regular s-blocks and hence its length does not exceed $6p_s$. (4.4) follows from the observation that a package of s-blocks may contain one irregular s-block near its end (created by the code ϕ_{s+1}) and at most one irregular s-block created in later codes. It is so, because the cuts in codes $\phi_{t'+1}$ $(s \le t' < t)$ fall in different $s+1$-blocks, hence in different packages of s-blocks. ∎

5. Applications

We can now draw several conclusions from the main theorem 4.1. Some of them are known, in which case we provide a new proof, some others seem to be new.

COROLLARY 5.1: (Jewett-Krieger, Rosenthal [R]) *If (X, Σ, μ, T) is a nonperiodic ergodic measure-preserving transformation, then it has a topological zero-dimensional strictly ergodic model (X, T).*

The above follows since any single nonperiodic ergodic measure is a face in the simplex of invariant measures of the universal zero-dimensional metrizable system (the full one-sided shift with the Cantor alphabet). Our main theorem produces a **uniquely ergodic** zero-dimensional topological model. A minimal such model is then obtained by Theorem 1 of [D2].

COROLLARY 5.2: (see Kornfeld-Ormes [K-O] for the invertible case) *If K is a simplex with countably many extreme points then an arbitrary assignment of nonperiodic ergodic systems to the extreme points of K can be realized in a minimal zero-dimensional topological dynamical system.*

Proof. It suffices to find a face in the simplex of invariant measures of the universal system which realizes the assignment. Then the applications of our Theorem 4.1 and of Theorem 1 in [D2] will complete the proof. We will be using the following fact: *every nonperiodic ergodic measure has isomorphic copies distributed densely over the simplex of invariant measures of the universal system.* The proof relies on standard coding techniques and we will leave it to the reader (for instance, in a uniquely ergodic model one may replace all sufficiently long t-blocks in a 1-1 manner by t-blocks from a selected small ball).

We are in a position to construct a face realizing the given countable assignment. Enumerate the extreme points of K by $e_1, e_2, \dots$. In the simplex of invariant measures of the universal system find an ergodic measure μ_1, isomorphic to what we want to assign to e_1. Next, for $d_1 = \mathrm{dist}(e_1, e_2)$, within the d_1-ball around μ_1 find an ergodic measure $\mu_2 \neq \mu_1$, isomorphic to what we want to assign to e_2. Inductively, let c_n be a point in the subsimplex of K spanned by $\{e_1, e_2, \dots, e_n\}$ closest to e_{n+1}, and let $d_n = \mathrm{dist}(c_n, e_{n+1})$. Let ν_n be the point corresponding to c_n in the subsimplex spanned by $\{\mu_1, \mu_2, \dots, \mu_n\}$ (corresponding means with the same coefficients). Within the d_n-ball around ν_n find an ergodic measure μ_{n+1} (different from $\mu_1, \dots, \mu_n$) isomorphic to what we want to assign to e_{n+1}. It is now elementary to see that the face spanned by $\{\mu_1, \mu_2, \dots\}$ is affinely homeomorphic to K and obviously it has the desired identity assignment on the extreme points. ∎

Corollary 5.3: (see [D1]) *For every metrizable Choquet simplex K, there exists a minimal zero-dimensional dynamical system (X, T) whose simplex of invariant measures is affinely homeomorphic to K.*

Without using [D1], this follows from: 1) the well-known fact that a metrizable simplex in which the extreme points are dense (there is unique, up to affine homeomorphism, such simplex called the **Pulsen simplex**) has faces affinely homeomorphic to all possible metrizable Choquet simplexes, 2) the existence of a nonperiodic zero-dimensional system whose simplex of invariant measures is the Pulsen simplex (for example, the direct product of the full shift on two symbols with an odometer), and 3) the application of Theorem 4.1 and of Theorem 1 of [D2].

To demonstrate new possibilities revealed by Theorem 4.1 (combined with theorem 1 of [D2]) we provide the following example, which, to our knowledge, was unknown.

Example 5.4: For any pair of positive numbers $a < b$ there exist a minimal zero-dimensional dynamical system whose all ergodic measures are one-sided (or two-sided, if one wishes) Bernoulli and form a topological arc parametrized by their entropies ranging linearly from a to b. Likewise, we can construct minimal models with all ergodic measures being Bernoulli and arranged topologically as any preassigned metrizable compact, and with entropy varying continuously following any preassigned positive continuous function on this compact.

The above follows immediately, because such an arc (or compact set) of Bernoulli measures is easily found in the simplex of invariant measures of a full (one or two-sided) shift over a finite or countable alphabet. The spanned simplex is then a face in the simplex of all invariant measures of the full shift. Because Bernoulli measures are nonperiodic we can thus obtain a minimal realization.

Remark 5.5: It is important to realize that two Choquet simplexes whose sets of extreme points are homeomorphic need not be affinely homeomorphic. The simplest example are simplexes with countably many extreme points arranged as a sequence μ_n converging to a nonextreme limit. In one of the simplexes, this limit can be equal to, say $1/2(\mu_1 + \mu_2)$, in another to $1/4\mu_1 + 3/4\mu_2$, in yet another to $1/3(\mu_1 + \mu_2 + \mu_3)$, etc. However, in the class of Bauer simplexes (i.e., with compact sets of extreme points), such condition suffices. We have implicitly used this in the example 5.4.

6. Questions

Let us verbalize some natural questions related to the topological or minimal realizability of the assignments. Most of these questions are implicit in [D2] or [K-O], but they deserve to be formulated and commented in one place in order to indicate possible directions of further investigations.

Question 6.1: Are the families of all topological and of all minimal assignments essentially larger than the families of topological zero-dimensional and of minimal zero-dimensional assignments, respectively?

Comment: There are many possibilities to replace a system by its zero-dimensional extension without changing the assignment (so-called **small boundary property**, see the work of E. Lindenstrauss [L]), and for many examples of higher-dimensional systems at least one of these possibilities is available (see [D2], the list preceding Theorem 2). But not for all of them. Examples are presented also in [L]. On the other hand, it seems that dimension zero imposes the weakest possible topological constraints, allowing the highest flexibility for realizations of measure-preserving systems. So, it is hard to expect that there exist assignments realizable only on some connected or partly connected spaces.

Question 6.2: Is Theorem 1 of [D2] true for any topological nonperiodic assignment (not necessarily zero-dimensional)?

Comment: This is automatically true if Question 6.1 has a negative answer. In [D2], Theorem 2, we indicate a class of higher-dimensional systems for which Theorem 1 still holds (extensions of systems with the small boundary property, in particular, extensions of zero-dimensional systems). Of course, there is no indication that the class of topological assignments on such spaces is essentially larger than that on zero-dimensional ones.

Question 6.3: Is Theorem 4.1 of this work true for any topological nonperiodic assignment (not necessarily zero-dimensional)?

Comment: Again, this is true if Question 6.1 is false. By the applied "marker methods", it is not hard to see that Theorem 4.1 extends to at least the same class of higher-dimensional systems as indicated in Theorem 2 of [D2].

Question 6.4: By a **factor** of an assignment Ψ on K we will mean an assignment Ψ' on some K', such that there exists an affine continuous surjection $\pi : K \to K'$ and for every $p \in K$ a factor map of measure preserving transformations from $\Psi(p)$ to $\Psi'(\pi(p))$. Is every factor of a topological (minimal) assignment topological (minimal)?

Comment: Every topological dynamical system admits a zero-dimensional extension. Its assignment is hence a factor (in the above sense) of the assignment of this extension. Thus, it suffices to answer the question for topological zero-dimensional assignments. If in addition, one could prove that a factor of such an assignment is again realizable in dimension zero, this would answer (negatively) the Question 6.1, hence resolve all the other questions formulated above; minimal assignments would coincide with topological nonperiodic assignments. This direction seems to be the most promising for further investigations.

Question 6.5: Is Theorem 4.1 true without assuming nonperiodicity of the face?

COMMENT: Then the composition with Theorem 1 of [D2] is not possible, but for Theorem 4.1 alone there are no immediate reasons why it should not hold. Of course the "marker methods" would have to be essentially modified, because periodic points do not posses t-markers for large t.

Question 6.6: ([K-O]) Is any topological nonperiodic (perhaps zero-dimensional) invertible assignment on a simplex K minimally realizable within the orbit equivalence class of an arbitrary Cantor minimal system whose simplex of invariant measures is affinely homeomorphic to K?

COMMENT: Answering this question requires refinements of the methods used in [K-O], allowing to deal with uncountably many ergodic measures. See [K-O] for more comments on such attempts.

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